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Integral relations and symmetry group expansions for the Helmholtz equation

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Abstract. New integral relations between the eigenfunctions (Bessel functions and Legendre functions) of the Helmholtz equation in spherical, cylindrical and Cartesian coordinate systems are obtained. Expansion of particular solutions in one coordinate system with respect to eigenfunctions of the same equation in another coordinate system is applied. A stationary phase method is used to find the expansion coefficients.

1. Introduction

Expansions in functions forming a base of symmetry group representation are widely exploited in quantum mechanics. We note their usefulness in finding very practical integral relations between mathematical functions with potentially wide applications in physics and technology.

The proposed method of finding such relations is applicable if the expansions contain a parameter able to attain large values and the expansion coefficients are calculated by the stationary phase method (or saddle-point method). As shown in the following, with the example of the Helmholtz equation, the functions depending on that parameter factorize. Due to factorization, the stationary phase method enables one to find *exact* expressions for the expansion coefficients.

2. Relations between eigenfunctions of the Helmholtz equation in different coordinate systems

The particular solutions of the Helmholtz equation (which is the same as the Schrödinger equation for free space) for three-dimensional infinite space

$$\Delta\Psi + k^2\Psi = 0$$

in the Cartesian (x, y, z), spherical (r, θ, φ) and cylindrical (ρ, z, φ) coordinate systems have the following forms:

$$\Psi_{k_x k_y k_z}(x, y, z) \propto \mathrm{e}^{\mathrm{i}(k_x x + k_y y + k_z z)} \qquad k_x^2 + k_y^2 + k_z^2 = k^2 \tag{1}$$

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$$\Psi_{\ell m}(r,\theta,\varphi) \propto \mathrm{e}^{\mathrm{i}m\varphi}\sqrt{k/r}\,Z_{\ell+1/2}(kr)\,P_\ell^m(\cos\theta) \tag{2}$$

$$\Psi_{mk_z}(\rho,z,\varphi) \propto \mathrm{e}^{\mathrm{i}(m\varphi+k_z z)}Z_m\big(\rho\sqrt{k^2-k_z^2}\big) \tag{3}$$

where $P_\ell^m(\cos\theta)$ is the associated Legendre function, Z is any Bessel function, $m = 0, 1\ldots\ell$; $\ell \in \mathbb{N}$, $r^2 = \rho^2 + z^2$, $\rho = r\sin\theta$, $z = r\cos\theta$, $\varphi = \operatorname{arccot}(x/y)$.

Functions (1)–(3) form the bases of the symmetry groups (translations, three-dimensional rotations and screw translations along the z-axis) representations for the Helmholtz equation. Each of these bases is a complete system. Any function can be expressed by one of the bases.

Let us expand the function $\Psi_{mk_z}(\rho,z,\varphi)$ (equation (3)) with the definite k and k_z in terms of functions $\Psi_{k_xk_yk_z}(x,y,z)$ (equation (1)). Since k and k_z are fixed the expansion must involve the variable k_x and certain $k_y^2 = k^2 - k_z^2 - k_x^2$:

$$\mathrm{e}^{\mathrm{i}m\varphi}H_m^{(j)}(\rho k_t) = \int_{-\infty}^{\infty}\mathrm{d}k_x\,A^{(j)}(k_x)\exp\Big[\mathrm{i}\big(k_x x \pm y\sqrt{k_t^2-k_x^2}\big)\Big] \tag{4}$$

where $k_t = \sqrt{k^2-k_z^2}$ and the sign will be determined later. The common phase factor $\exp(\mathrm{i}k_z z)$ is omitted. The Hankel functions $H_m^{(j)}(z)$ $(j = 1, 2)$ are convenient to begin with, since the remaining Bessel functions can be expressed by their linear combinations. The coefficients $A^{(j)}$ are found below (see equations (13) and (15)). Substituting these expressions into (4) one can see, that the integral (4) converges absolutely.

The expansion of the function (2) in terms of functions (3) takes the form

$$\sqrt{k/r}\,H_{\ell+1/2}^{(j)}(kr)\,P_\ell^m(z/r) = \int_{-\infty}^{\infty}\mathrm{d}k_z\,B^{(j)}(k_z)\,H_m^{(j)}\big(\rho\sqrt{k^2-k_z^2}\big)\exp(\mathrm{i}k_z z) \tag{5}$$

where the phase factor $\exp(\mathrm{i}m\varphi)$ is disregarded. The Hankel functions appearing on both sides of (5) bear the same upper index j, as follows from their behaviour at $\rho \to \infty$.

The expansion of the function (2) in terms of functions (1) involves two variables, k_x and k_y:

$$\sqrt{k/r}\,\mathrm{e}^{\mathrm{i}m\varphi}H_{\ell+1/2}^{(j)}(kr)\,P_\ell^m(z/r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}k_x\,\mathrm{d}k_y\,C^{(j)}(k_x,k_y)\exp\Big[\mathrm{i}\big(k_x x + k_y y \pm z\sqrt{k^2-k_x^2-k_y^2}\big)\Big]. \tag{6}$$

We look for functions $A^{(j)}(k_x)$, $B^{(j)}(k_z)$ and $C^{(j)}(k_x,k_y)$. Since the identities (4), (5) and (6) are fulfilled for arbitrary x, z and (x, y), respectively, one can express their Fourier transforms as

$$2\pi A^{(j)}(k_x)\exp\big(\pm\mathrm{i}y\sqrt{k_t^2-k_x^2}\big) = \int_{-\infty}^{\infty}\mathrm{d}x\,H_m^{(j)}(\rho k_t)\exp[\mathrm{i}(-k_x x + m\varphi)] \tag{7}$$

$$2\pi B^{(j)}(k_z)H_m^{(j)}\big(\rho\sqrt{k^2-k_z^2}\big) = \int_{-\infty}^{\infty}\mathrm{d}z\,\sqrt{k/r}\,H_{\ell+1/2}^{(j)}(kr)\,P_\ell^m(z/r)\exp(-\mathrm{i}k_z z) \tag{8}$$

$$(2\pi)^2 C^{(j)}(k_x, k_y) \exp\left(\pm \mathrm{i} z\sqrt{k^2 - k_x^2 - k_y^2}\right)$$

$$= \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{d}y\, \sqrt{k/r}\, H^{(j)}_{\ell+1/2}(kr) P^m_\ell(z/r) \exp[-\mathrm{i}(k_x x + k_y y - m\varphi)]. \tag{9}$$

In equations (7)–(9) $\rho = \sqrt{x^2 + y^2}$, $r = \sqrt{z^2 + \rho^2}$ and $\varphi = \operatorname{arccot}(x/y)$.

Let us emphasize that the integral (7) at $y \neq 0$ exists as a limiting expression:

$$\int\limits_{-\infty}^{\infty} \mathrm{d}x(\cdots) = \lim_{R_1, R_2 \to \infty} \int\limits_{-R_1}^{R_2} \mathrm{d}x(\cdots)$$

because $H^{(j)}_m \simeq \rho^{-1/2}$ at $\rho \to \infty$. Integrals (8) and (9) have the similar sense.

Equations (7)–(9) contain parameters y, ρ and z, which may be large. Thus one can compare both sides of these equations at large values of parameters and obtain *exact* expressions for $A^{(j)}$, $B^{(j)}$ and $C^{(j)}$, because the latter do not depend on these parameters. The procedure is successful, since the integrals (7)–(9) can be evaluated by the stationary phase method in this limit.

In equation (7) for $yk_t \gg 1$ one can apply the asymptotic representation [1]:

$$H^{(1,2)}_m(z) \simeq \left(\frac{2}{\pi z}\right)^{1/2} \exp\left[\pm \mathrm{i}\left(z - m\frac{\pi}{2} - \frac{\pi}{4}\right)\right] \tag{10}$$

where the sign $+(-)$ corresponds to the superscript 1(2) in the Hankel function. After substitution $x = y\cot\varphi$ and (10) into the right-hand side of (7) we can write the integral on the right-hand side of (7) for $y > 0$ in the following form:

$$\sqrt{\frac{2y}{\pi k_t}} \exp\left[\mp \mathrm{i}\left(m\frac{\pi}{2} + \frac{\pi}{4}\right)\right] \int\limits_0^{\pi} \frac{\mathrm{d}\varphi}{\sin^{3/2}\varphi} \exp\{\mathrm{i}[y f(\varphi) + m\varphi]\} \tag{11}$$

with

$$f(\varphi) = \pm\frac{k_t}{\sin\varphi} - k_x \cot\varphi\,.$$

For $k_x^2 < k_t^2$, the phase function $f(\varphi)$ in the interval $[0, \pi]$ has one stationary point at $\cos\varphi_0 = \pm k_x/k_t$ and $f(\varphi_0) = \pm(k_t^2 - k_x^2)^{1/2}$; $(\mathrm{d}^2 f/\mathrm{d}\varphi^2)_{\varphi=\varphi_0} = \pm k_t[1 - (k_x/k_t)^2]^{-1/2}$. Taking the integral in (11) by the stationary phase method we obtain

$$\frac{2\,\mathrm{i}^{-m}}{\sqrt{k_t^2 - k_x^2}} \left[1 + \mathrm{O}(1/yk_t)\right] \exp\left[\pm \mathrm{i}\left(y\sqrt{k_t^2 - k_x^2} + m \arccos\frac{k_x}{k_t}\right)\right]. \tag{12}$$

The next term in the asymptotic expansion (10) is of the order of z^{-1}, that is $(yk_t)^{-1}\sin\varphi$ in variables of the integral (11). Therefore this expansion is uniformly asymptotic with respect to φ at $yk_t \gg 1$ and can be integrated.

Matching the leading term in (12) at $yk_t \gg 1$ with the left-hand side of (7) gives

$$A^{(1,2)}(k_x) = \frac{\mathrm{i}^{-m}}{\pi\sqrt{k_t^2 - k_x^2}} \exp\left(\pm \mathrm{i} m \arccos\frac{k_x}{k_t}\right) \tag{13}$$

where the signs $\pm$ correspond to the superscripts $(1, 2)$.

Hence, we write (7) for $k_x^2 < k_t^2$ in the form

$$\int_{-\infty}^{\infty} \mathrm{d}x\, H_m^{(j)}(\rho k_t) \exp[\mathrm{i}(-k_x x + m\varphi)] = \frac{2\,\mathrm{i}^{-m}}{\sqrt{k_t^2 - k_x^2}} \exp\left[\pm \mathrm{i}\left(y\sqrt{k_t^2 - k_x^2} + m \arccos\frac{k_x}{k_t}\right)\right]. \tag{14}$$

For $k_x^2 > k_t^2$, the stationary point is displaced from the real axis in the complex plane φ and we obtain

$$A^{(1,2)}(k_x) = \frac{\mp \mathrm{i}^{1-m}}{\pi\sqrt{k_x^2 - k_t^2}} \exp\left(-m \operatorname{sgn} k_x\ \operatorname{arch}\frac{|k_x|}{k_t}\right) \tag{15}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, H_m^{(j)}(\rho k_t) \exp[\mathrm{i}(-k_x x + m\varphi)] = \frac{\mp 2\,\mathrm{i}^{1-m}}{\sqrt{k_x^2 - k_t^2}} \exp\left(-y\sqrt{k_x^2 - k_t^2} - m \operatorname{sgn} k_x\ \operatorname{arch}\frac{|k_x|}{k_t}\right). \tag{16}$$

Integrals (14) and (16) may be rewritten in the limits 0 and ∞, since $\varphi = \operatorname{arccot}(x/y) = \pi/2 - \arcsin(x/\rho)$.

By using the relations [1]:

$$H_m^{(1)}(z) + H_m^{(2)}(z) = 2J_m(z) \qquad H_m^{(1)}(z) - H_m^{(2)}(z) = 2\mathrm{i}Y_m(z) \tag{17}$$

we obtain from (14) and (16)

$$\int_0^{\infty} \mathrm{d}x\, J_m(\rho k_t) \cos[k_x x + m \arcsin(x/\rho)]$$
$$= \begin{cases} \dfrac{(-1)^m}{\sqrt{k_t^2 - k_x^2}} \cos\left(y\sqrt{k_t^2 - k_x^2} + m \arccos\dfrac{k_x}{k_t}\right) & \text{for } k_x^2 < k_t^2 \\ 0 & \text{for } k_x^2 > k_t^2 \end{cases} \tag{18}$$

$$\int_0^{\infty} \mathrm{d}x\, Y_m(\rho k_t) \cos\left[k_x x + m \arcsin(x/\rho)\right]$$
$$= \begin{cases} \dfrac{(-1)^m}{\sqrt{k_t^2 - k_x^2}} \sin\left(y\sqrt{k_t^2 - k_x^2} + m \arccos\dfrac{k_x}{k_t}\right) & \text{for } k_x^2 < k_t^2 \\ \dfrac{(-1)^{m+1}}{\sqrt{k_x^2 - k_t^2}} \exp\left(-y\sqrt{k_x^2 - k_t^2} - m \operatorname{sgn} k_x \operatorname{arch}\dfrac{|k_x|}{k_t}\right) & \text{for } k_x^2 > k_t^2\,. \end{cases} \tag{19}$$

In equations (14)–(19) $y > 0$.

The expression (4) with $A^{(j)}$ defined by (13) leads to the known representation of the Hankel functions in the form of a contour integral (see, for example, [2] 8.423). Integrals (18) and (19) have been known up until now for $m = 0$ only (see 1.12 (37), (41) in [3]).

After the substitution $z = \rho \cot\theta$ we obtain for the right-hand side of (8)

$$\int_0^{\pi} \mathrm{d}\theta\, \frac{\sqrt{k\rho}}{\sin^{3/2}\theta} H_{\ell+1/2}^{(j)}\left(\frac{k\rho}{\sin\theta}\right) P_\ell^m(\cos\theta) \exp(-\mathrm{i}k_z\rho\cot\theta)\,.$$

For $k\rho \gg 1$, the asymptotic behaviour described in (10), leads us to the integral

$$\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{\mp \mathrm{i}\pi(\ell+1)/2}\int_0^\pi \frac{\mathrm{d}\theta}{\sin\theta}\,P_\ell^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}\rho f(\theta)}$$

with

$$f(\theta) = \pm\frac{k}{\sin\theta} - k_z\cot\theta$$

which is the same function, as in (11). After taking the integral by the stationary phase method we find the main contribution

$$\frac{2}{\sqrt{\rho}(k^2-k_z^2)^{1/4}}\,P_\ell^m\left(\pm\frac{k_z}{k}\right)\exp\left[\pm\mathrm{i}\left(\rho\sqrt{k^2-k_z^2}-\frac{\ell\pi}{2}-\frac{\pi}{4}\right)\right].$$

Comparing the last expression with the asymptotic representation of the left-hand side of (8) and applying the relation $P_\ell^m(-x) = (-1)^{\ell-m}P_\ell^m(x)$, we write

$$B^{(j)}(k_z) = \frac{\mathrm{i}^{m-\ell}}{\sqrt{2\pi}}\,P_\ell^m\left(\frac{k_z}{k}\right). \tag{20}$$

Hence, we obtain the integral relating the spherical and cylindrical Hankel functions:

$$\int_{-\infty}^{\infty}\mathrm{d}z\,\sqrt{\frac{k}{2\pi r}}\,H_{\ell+1/2}^{(j)}(kr)\,P_\ell^m\left(\frac{z}{r}\right)\mathrm{e}^{-\mathrm{i}k_z z} = \mathrm{i}^{m-\ell}H_m^{(j)}\big(\rho\sqrt{k^2-k_z^2}\big)\,P_\ell^m\left(\frac{k_z}{k}\right) \tag{21}$$

where $\rho > 0$. The formula is valid if $\pi > \arg\sqrt{k^2-k_z^2} \geqslant 0$ for $j = 1$ and $-\pi < \arg\sqrt{k^2-k_z^2} \leqslant 0$ for $j = 2$. In particular, for $\ell = m$ the integrals (21) are given in [3] (see 1.13(42)).

Now, we rewrite the right-hand side of (21) in an explicit form for the case $k^2 - k_z^2 = \zeta < 0$. As it is evident from (21), passing from $\zeta > 0$ to $\zeta < 0$ must be performed in the complex plane $\pi > \arg\sqrt{\zeta} \geqslant 0$ for the case of $j = 1$ and in the $-\pi < \arg\sqrt{\zeta} \leqslant 0$ one for the case of $j = 2$. By making use of the definition

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}(x^2-1)^\ell}{\mathrm{d}x^{\ell+m}} \tag{22}$$

one obtains the following relations:

$$H_m^{(1)}(\mathrm{i}x) = \frac{2}{\pi}\mathrm{i}^{-m-1}K_m(x) \qquad P_\ell^m\left(\frac{k_z}{k}\right) = \mathrm{i}^m\mathcal{P}_\ell^m\left(\frac{k_z}{k}\right)$$

for the former case and

$$H_m^{(2)}(-\mathrm{i}x) = \frac{2}{\pi}\mathrm{i}^{m+1}K_m(x) \qquad P_\ell^m\left(\frac{k_z}{k}\right) = \mathrm{i}^{-m}\mathcal{P}_\ell^m\left(\frac{k_z}{k}\right)$$

for the latter case, where K_m is the Macdonald function and $\mathcal{P}_\ell^m(x)$ takes on real values for $|x| > 1$:

$$\mathcal{P}_\ell^m(x) = \frac{(-1)^m}{2^\ell\ell!}(x^2-1)^{m/2}\frac{\mathrm{d}^{\ell+m}(x^2-1)^\ell}{\mathrm{d}x^{\ell+m}}.$$

We rewrite (21) for $k_z^2 - k^2 > 0$ in the form

$$\int_{-\infty}^{\infty}\mathrm{d}z\,\sqrt{\frac{k}{2\pi r}}\,H_{\ell+1/2}^{(1,2)}(kr)\,P_\ell^m\left(\frac{z}{r}\right)\mathrm{e}^{-\mathrm{i}k_z z} = \mp\frac{2}{\pi}\mathrm{i}^{m-\ell+1}K_m\big(\rho\sqrt{k_z^2-k^2}\big)\mathcal{P}_\ell^m\left(\frac{k_z}{k}\right). \tag{23}$$

For the special case of $\ell = m = 0$ integrals (21), (23) are already known (see, 6.616 (3,4) in [2]).

Let us add (21) (and equation (23)) for $j = 1$ to (21) (and equation (23)) for $j = 2$ (see equation (17)). There follows an integral relation between the Bessel functions [4]:

$$\int_{-\infty}^{\infty} \mathrm{d}z \sqrt{\frac{k}{2\pi r}} J_{\ell+1/2}(kr) P_\ell^m \left(\frac{z}{r}\right) \mathrm{e}^{-\mathrm{i}k_z z}$$
$$= \begin{cases} \mathrm{i}^{m-\ell} J_m\big(\rho\sqrt{k^2 - k_z^2}\big) P_\ell^m \left(\dfrac{k_z}{k}\right) & \text{for } k_z^2 < k^2 \\ 0 & \text{for } k_z^2 > k^2 \;. \end{cases} \tag{24}$$

If we put $\ell = m$ in (24) we obtain the integral 2.12.22(7) in [5] (see also 1.13(37) in [3]). If we subtract (21) (and (23)) for $j = 2$ from (21) (and (23)) for $j = 1$ (see (17)), we obtain similar integral relation between the Neumann functions:

$$\int_{-\infty}^{\infty} \mathrm{d}z \sqrt{\frac{k}{2\pi r}} Y_{\ell+1/2}(kr) P_\ell^m \left(\frac{z}{r}\right) \mathrm{e}^{-\mathrm{i}k_z z}$$
$$= \begin{cases} \mathrm{i}^{m-\ell} Y_m\big(\rho\sqrt{k^2 - k_z^2}\big) P_\ell^m \left(\dfrac{k_z}{k}\right) & \text{for } k_z^2 < k^2 \\ -\dfrac{2}{\pi}\mathrm{i}^{m-\ell} K_m\big(\rho\sqrt{k_z^2 - k^2}\big) \mathcal{P}_\ell^m \left(\dfrac{k_z}{k}\right) & \text{for } k_z^2 > k^2 \,. \end{cases} \tag{25}$$

In equations (23)–(25) $\rho > 0$. The integrals 2.13.8(14) in [5], 1.13(41) in [3] and 7.230 in [6] are the particular (for $\ell = m$) cases of our expression (25).

Equation (5) with $B^{(j)}$ in form of (20) takes the form

$$\int_{-\infty}^{\infty} \mathrm{d}k_z\, H_m^{(j)}\big(\rho\sqrt{k^2 - k_z^2}\big) P_\ell^m \left(\frac{k_z}{k}\right) \mathrm{e}^{\mathrm{i}k_z z} = \mathrm{i}^{\ell-m}\sqrt{\frac{2\pi k}{r}} H_{\ell+1/2}^{(j)}(kr) P_\ell^m \left(\frac{z}{r}\right) \tag{26}$$

where $\pi > \arg\sqrt{k^2 - k_z^2} \geqslant 0$ for $j = 1$ and $-\pi < \arg\sqrt{k^2 - k_z^2} \leqslant 0$ for $j = 2$. The integrals 1.13(58), 1.13(59) in [3] and 2.14.2(4), 2.14.2(9) in [5] are particular ($\ell = m$) cases of (26).

The inverse Fourier transform corresponding to (24) and (25) bears the form

$$\int_{-k}^{k} \mathrm{d}k_z\, J_m\big(\rho\sqrt{k^2 - k_z^2}\big) P_\ell^m \left(\frac{k_z}{k}\right) \mathrm{e}^{\mathrm{i}k_z z} = \mathrm{i}^{\ell-m}\sqrt{\frac{2\pi k}{r}} J_{\ell+1/2}(kr) P_\ell^m \left(\frac{z}{r}\right) \tag{27}$$

$$\sqrt{\frac{2\pi k}{r}} Y_{\ell+1/2}(kr) P_\ell^m \left(\frac{z}{r}\right) = \mathrm{i}^{m-\ell} \int_{-k}^{k} \mathrm{d}k_z\, Y_m\big(\rho\sqrt{k^2 - k_z^2}\big) P_\ell^m \left(\frac{k_z}{k}\right) \mathrm{e}^{\mathrm{i}k_z z}$$
$$- \frac{4}{\pi}\int_{k}^{\infty} \mathrm{d}k_z\, \cos[k_z z + \pi(m-\ell)/2] \mathcal{P}_\ell^m \left(\frac{k_z}{k}\right) K_m\big(\rho\sqrt{k_z^2 - k^2}\big) \,. \tag{28}$$

The special cases of integral (27) for $\ell = m$ are presented in [3] 1.13(50), 8.7(48) (see also integrals 2.12.21(5) in [5] and 7.219 in [6]). The integrals 7.302 in [6] and 2.13.8(13) in [5] are particular cases (for $\ell = m$) of expression (28).

In the expression (9) we introduce spherical angles ϑ and ψ of vector $\boldsymbol{k}$ and variables ρ and φ instead of x, y. Hence, the integral in (9) takes on the form

$$\mathrm{e}^{\mathrm{i}m(\psi-\pi/2)}\sqrt{k}\int_0^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2+z^2)^{1/4}} H^{(j)}_{\ell+1/2}(kr)P_\ell^m(z/r)\int_{-\pi}^{\pi}\mathrm{d}\varphi\,\mathrm{e}^{-\mathrm{i}(m\varphi-k\rho\sin\varphi\sin\vartheta)}.$$

The integral over φ generates the Bessel function. Let us go from integrating over $\rho = z\tan\theta$ to integrating over the variable θ, which belongs to the interval $[0, \pi/2]$ for $z > 0$. We obtain the right-hand side of (9) in the form

$$2\pi k^{1/2}z^{3/2}\mathrm{e}^{\mathrm{i}m(\psi-\pi/2)}\int_0^{\pi/2}\frac{\sin\theta\,\mathrm{d}\theta}{\cos^{5/2}\theta}H^{(j)}_{\ell+1/2}\left(\frac{kz}{\cos\theta}\right)P_\ell^m(\cos\theta)J_m(kz\tan\theta\sin\vartheta)\,.$$

For large argument values one can again apply the asymptotic representation of Bessel functions. Taking the latter integral by the stationary phase method we obtain

$$\frac{2\sqrt{2\pi}}{k\cos\vartheta}P_\ell^m(\cos\vartheta)\exp\left[\mathrm{i}\left(m\psi - m\frac{\pi}{2}\pm kz\cos\vartheta\mp\ell\frac{\pi}{2}\pm m\frac{\pi}{2}\right)\right]$$

where $\cos\vartheta = \sqrt{1-(k_t/k)^2}$. On applying (9) we find

$$C^{(1,2)}(k_x,k_y) = \frac{2}{(2\pi)^{3/2}k\cos\vartheta}P_\ell^m(\cos\vartheta)\exp\left[\mathrm{i}\left(m\psi - m\frac{\pi}{2}\mp\ell\frac{\pi}{2}\pm m\frac{\pi}{2}\right)\right]. \tag{29}$$

Thus, one can write (9) in the form

$$\int_0^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2+z^2)^{1/4}}H^{(j)}_{\ell+1/2}(kr)P_\ell^m(z/r)J_m(\rho k_t)$$
$$= \frac{\mathrm{i}^{\pm(m-\ell)}\sqrt{2/\pi}}{k^{3/2}\cos\vartheta}P_\ell^m(\cos\vartheta)\exp(\pm\mathrm{i}kz\cos\vartheta) \tag{30}$$

where $\cos\vartheta = \sqrt{1-(k_t/k)^2}$, $z > 0$. The formula (30) is valid if $\pi > \arg\sqrt{k^2-k_t^2} \geqslant 0$ for $j = 1$ and $-\pi < \arg\sqrt{k^2-k_t^2} \leqslant 0$ for $j = 2$. We can continue analytically the functions in (30) for the case $k_t > k$. Equation (22) gives

$$P_\ell^m(\pm\mathrm{i}x) = \Pi_\ell^m(x)\mathrm{i}^{\mp(\ell+m)}$$

where

$$\Pi_\ell^m(x) = \frac{(-1)^{\ell+m}}{2^\ell\ell!}(1+x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}(x^2+1)^\ell}{\mathrm{d}x^{\ell+m}}\,.$$

Thus we have for $k_t > k$

$$\int_0^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2+z^2)^{1/4}}H^{(1,2)}_{\ell+1/2}(kr)P_\ell^m(z/r)J_m(\rho k_t)$$
$$= \mp\,\mathrm{i}(-1)^\ell\frac{\sqrt{2/\pi k}}{\sqrt{k_t^2-k^2}}\Pi_\ell^m\left(\sqrt{(k_t/k)^2-1}\right)\exp\left(-z\sqrt{k_t^2-k^2}\right). \tag{31}$$

On applying (30) and (31), we can write

$$\int_0^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2+z^2)^{1/4}} J_{\ell+1/2}(kr) P_\ell^m(z/r) J_m(\rho k_t)$$
$$= \begin{cases} \dfrac{\sqrt{2/\pi}}{k^{3/2}\cos\vartheta} P_\ell^m(\cos\vartheta)\cos\left[kz\cos\vartheta + \dfrac{\pi}{2}(m-\ell)\right] & \text{for } k_t^2 < k^2 \\ 0 & \text{for } k_t^2 > k^2 \end{cases} \tag{32}$$

$$\int_0^\infty \frac{\rho\,\mathrm{d}\rho}{(\rho^2+z^2)^{1/4}} Y_{\ell+1/2}(kr) P_\ell^m(z/r) J_m(\rho k_t)$$
$$= \begin{cases} \dfrac{\sqrt{2/\pi}}{k^{3/2}\cos\vartheta} P_\ell^m(\cos\vartheta)\sin\left[kz\cos\vartheta + \dfrac{\pi}{2}(m-\ell)\right] & \text{for } k_t^2 < k^2 \\ \dfrac{(-1)^{\ell+1}\sqrt{2/\pi k}}{\sqrt{k_t^2-k^2}} \Pi_\ell^m\left(\sqrt{(k_t/k)^2-1}\right)\exp\left(-z\sqrt{k_t^2-k^2}\right) & \text{for } k_t^2 > k^2 \end{cases} \tag{33}$$

where $\cos\vartheta = \sqrt{1-(k_t/k)^2},\ z > 0$.

The special case of the integral (32) for $\ell = m$ can be found in [2] (see 6.596 (6)). The integral (6) with $C^{(j)}$ given by (29) can be reduced to (5).

3. Summary

The most essential results of this work represent the pairs of mutual relations between: (i) the cylindrical and spherical Bessel functions of the first kind (equations (24) and (27)), (ii) the cylindrical and spherical Bessel functions of the second kind (Neumann functions, equations (25) and (28)), (iii) the cylindrical and spherical Bessel functions of the third kind (Hankel functions, equations (21) and (26)).

We also find the interesting integrals of Bessel functions (18), (19) expressed as elementary functions.

The relations found between the cylindrical and spherical Bessel, Neumann and Hankel functions are of special importance in a wide class of physical problems concerning finite-dimensional systems and electron dynamics considered in [7]. The functions appearing in the integral relations derived in this paper depend on many parameters. As a result, we are led to a remarkable extension of the existing collections on definite integrals. In available sources only a few particular cases of the integrals obtained are presented. These special cases of our relations are cited in the preceding text. The integral (27) expressed in a more complicated form is known in the particular case of $k = 1$ (see [8]). The parameter k can be eliminated from (27) by the following rescaling procedure: $k_z/k \to p',\ kr \to r',\ k\rho \to \rho',\ kz \to z'$. Nevertheless, keeping the spectral parameter k in these expressions is essential for practical uses. If we substitute $k = 1,\ p = \cos u,\ \rho = r\sin\theta$ and $z = r\cos\theta$ into (27) we obtain the integral given in [8] (see chapter 10 at the end of the section on Bessel functions).

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